

On the minimal length of extremal rays for Fano 4-folds

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Abstract

The minimum of intersection numbers of the anti-canonical divisor with rational curves on a Fano manifold is called pseudo-index. It is expected that the intersection number of anti-canonical divisor attains to the minimum on an extremal ray, i.e. there exists an extremal rational curve whose intersection number with the anti-canonical divisor equals the pseudo-index. In this note, we prove this for smooth Fano 4-folds having birational contractions.

1 Introduction

Let X be a Fano manifold, i.e. a smooth projective variety such that the anti-canonical divisor $-K_X$ is ample. Throughout this paper, algebraic varieties are defined over the field of complex numbers.

The *index* $r(X)$ and the *pseudo-index* $i(X)$ of a Fano manifold X is defined respectively as

$$r(X) := \max\{m \in \mathbb{N} \mid -K_X = mH \text{ for some } H \in \text{Pic}(X)\},$$

$$i(X) := \min\{-K_X \cdot \Gamma \mid \Gamma \text{ is a rational curve on } X\}.$$

By definition, the positive integer $i(X)$ is a multiple of $r(X)$. The equality $r(X) = i(X)$ does not hold in general. For example, if $X = \mathbb{P}^a \times \mathbb{P}^b$, then we have $r(X) = \gcd(a+1, b+1)$ while $i(X) = \min(a+1, b+1)$. On the other hand, when $\rho(X) = 1$, we do not know whether the equality holds or not (see [9] p. 248 Problem 1.13).

In [19], J. A. Wiśniewski observed that the pseudo-index is well adapted to the study of Fano manifolds with Picard number greater than or equal to 2, since it is used to give a lower bound for the dimension of the deformation space of rational curves. However, in view of the fact that the geometric structure of Fano manifolds is governed by its extremal rays, it is essential to consider not all rational curves but only the extremal rational curves. So, we define another invariant $\ell(X)$ as follows.

Recall that the *length* of an extremal ray $R \subset \overline{\text{NE}}(X)$ is defined by

$$\ell(R) := \min\{-K_X \cdot \Gamma \mid [\Gamma] \in R\}.$$

Note that the Kleiman-Mori cone of a Fano manifold is generated by a finite number of extremal rays. We define the *minimal length* of extremal rays for a Fano manifold X as

$$\ell(X) := \min\{\ell(R) \mid R \text{ is an extremal ray of } \overline{\text{NE}}(X)\}$$

That is, the positive integer $\ell(X)$ is the minimal anti-canonical degree among all extremal rational curves on X . Clearly, we have $\ell(X) \geq i(X)$. A natural problem is the following:

Problem: Do we have $i(X) = \ell(X)$ for any Fano manifold X ?

The purpose of this note is to give an affirmative answer in a special case:

Theorem 1. *Let X be a smooth Fano 4-fold. Assume that X has a birational contraction. Then, we have $i(X) = \ell(X)$.*

Note that in some cases, the equality $i(X) = \ell(X)$ is easily verified:

- When $\rho(X) = 1$, the equality is obvious, since (the numerical class of) any curve on X generates the extremal ray.
- If there is an extremal ray $R \subset \overline{\text{NE}}(X)$ such that $\ell(R) = 1$, then clearly $i(X) = \ell(X) = 1$.
- If X is a toric Fano manifold, the equality follows from the fact that any curve on X is numerically equivalent to a linear combination of T -invariant curves with natural number coefficients (see the proof of Proposition 2.26 in [14]).

Remark. Concerning the last observation on the toric case, a similar statement is expected in general. For simplicity, we consider a Fano manifold X with $\rho(X) = 2$. Then the Kleiman-Mori cone is generated by two extremal rays:

$$\overline{\text{NE}}(X) = R_1 + R_2.$$

Let f_i be the minimal rational curve of the extremal ray R_i , i.e. we assume that $-K_X \cdot f_i = \ell(R_i)$ for $i = 1, 2$.

Question: Let Γ be an irreducible curve on X . Do there exist positive integers a_1 and a_2 such that the 1-cycle $a_1 f_1 + a_2 f_2$ is numerically equivalent to Γ ?

The affirmative answer gives the equality $i(X) = \ell(X)$. Indeed, if $\Gamma_0 \subset X$ is a rational curve such that $-K_X \cdot \Gamma_0 = i(X)$, we write $\Gamma_0 \equiv a_1 f_1 + a_2 f_2$ with $a_1, a_2 \in \mathbb{N}$, and we get

$$i(X) = -K_X \cdot \Gamma_0 = a_1(-K_X \cdot f_1) + a_2(-K_X \cdot f_2) \geq \min\{\ell(R_1), \ell(R_2)\} = \ell(X).$$

In dimension three, the answer to the question is affirmative by [13] Proposition 6. The proof depends on numerical arguments on the intersection numbers of divisors on 3-folds, and seems difficult to apply it to higher dimensions. In this note, we treat only the problem of the equality $i(X) = \ell(X)$ in a direct way using classification results of Fano 4-folds.

The present note is organized as follows. Section 2 is devoted to show a preliminary result based on the bend-and-break lemma. In Section 3, we prove the equality $i(X) = \ell(X)$ when X has a birational contraction sending a divisor to a point. Section 4 gives a partial classification of Fano manifolds with $\ell(X) \geq 2$, which is necessary to our purpose. The proof of Theorem 1 is done in Section 5 using the results of Section 3 and 4.

Notation and conventions. The blow-up of a variety Y along a subvariety C is denoted by $\text{Bl}_C(Y)$. We denote by Q_k a smooth hyperquadric in \mathbb{P}^{k+1} . For a Cartier divisor E on a variety X and an extremal ray $R \subset \overline{\text{NE}}(X)$, the notation $E \cdot R > 0$ means that $E \cdot \alpha > 0$ for some $\alpha \in R$ (hence for any $\alpha \in R \setminus \{0\}$). For a vector bundle \mathcal{E} , we denote $\mathbb{P}(\mathcal{E})$ the Grothendieck's projectivization.

2 Unsplit family of rational curves

For the classification of Fano manifolds, it is important to compute the intersection number of extremal rational curves with special divisors. In this section, we prove a proposition on the intersection of rational curves with the exceptional divisor of a divisorial contraction.

The following lemma is well known but we include the proof for the reader's convenience.

Lemma 1. *Let $q : S \rightarrow B$ be a ruled surface over an irreducible curve B . Assume that there exists a morphism $p : S \rightarrow S'$ such that $\dim S' = 2$. Let D be an effective divisor on S . If $\dim p(\text{Supp}(D)) = 0$, then $\text{Supp}(D)$ is a section of q .*

Proof. (see [12] p.599, [6] p.460, [19] p.138, or [9] Ch.II. 5) We may assume that B is smooth (if B is singular, we consider its normalization $\tilde{B} \rightarrow B$ and the fiber product $\tilde{S} := S \times_B \tilde{B}$). Following the notation of [5] Ch.V. Proposition 2.8, let C_0 be a section of q such that $C_0^2 = -e$ and let f be a fiber of q .

Step 1. We show that D is irreducible. If not, let A_1 and A_2 be distinct irreducible components of $\text{Supp}(D)$. Since A_i is an exceptional curve, we have $A_i^2 < 0$ ($i = 1, 2$). Since $A_1 \neq A_2$, we have $A_1 \cdot A_2 \geq 0$. We write

$$A_i \equiv a_i C_0 + b_i f \quad (i = 1, 2).$$

Note that $a_i = A_i \cdot f \geq 1$ ($i = 1, 2$). We have $a_2 A_1 - a_1 A_2 \equiv (a_2 b_1 - a_1 b_2) f$, thus $(a_2 A_1 - a_1 A_2)^2 = 0$. Hence

$$0 \leq 2a_1 a_2 (A_1 \cdot A_2) = a_2^2 A_1^2 + a_1^2 A_2^2 < 0,$$

a contradiction. We conclude that D is irreducible.

Step 2. We show that $\Gamma := \text{Supp}(D)$ is a section. We write $\Gamma \equiv a C_0 + b f$. We consider the case $C_0^2 \leq 0$. We have $(\Gamma - a C_0)^2 = (b f)^2 = 0$. It follows that

$$2a(\Gamma \cdot C_0) = \Gamma^2 + a^2 C_0^2 < 0.$$

Hence, $\Gamma \cdot C_0 < 0$, which implies that $\Gamma = C_0$ is a section. Now, consider the case $C_0^2 > 0$. Assume that $a = \Gamma \cdot f \geq 2$. Then, by [5] Ch.V. Proposition 2.21 (a), we have $2b - ae \geq 0$. Hence,

$$0 \leq a(2b - ae) = (a C_0 + b f)^2 = \Gamma^2 < 0,$$

a contradiction. Therefore, $\Gamma \cdot f = 1$. □

We recall some notation on the family of rational curves from [9] to which we refer the reader for details. A *family of rational curves* on a projective variety X is an irreducible component V of the scheme $\text{RatCurves}^n(X)$ parameterizing rational curves on X . If V is proper, it is called *unsplit*. Let U be the universal family over V . Then we have the following basic diagram:

$$\begin{array}{ccc} U & \xrightarrow{p} & X \\ q \downarrow & & \\ & & V \end{array} \quad (1)$$

where q is a \mathbb{P}^1 -bundle and p is the map induced by the evaluation map. For $v \in V$, we denote by f_v the corresponding rational curve, i.e. $f_v := p(q^{-1}(v))$.

Proposition 1. *Let $\pi : X \rightarrow Y$ be the blow-up of a smooth projective variety Y of dimension ≥ 3 along a smooth curve C . Let E be the exceptional divisor. Let V be an unsplit family of rational curves on X such that $E \cdot f > 0$ for some (hence for any) $[f] \in V$. If $\dim V \geq 3$, then we have $\sharp(E \cap f) = 1$ for any $[f] \in V$ such that $f \not\subset E$.*

Proof. Consider the above diagram (1) of the family V . For a point $c \in C$ such that $p(U) \cap E_c \neq \emptyset$, we put:

$$E_c := \pi^{-1}(c), \quad U_c := p^{-1}(E_c), \quad V_c := q(U_c).$$

From the assumption that $E \cdot f > 0$, the rational curve f is not contracted by π . Thus, $q|_{U_c} : U_c \rightarrow V_c$ is a finite map. In particular, $\dim U_c = \dim V_c$. Hence, we have

$$\begin{aligned} \dim V_c &= \dim U_c \\ &\geq \dim U - \dim p(U) + \dim(p(U) \cap E_c) \\ &\geq \dim U - \dim p(U) + (\dim p(U) + \dim E_c - \dim X) \\ &= \dim U - 2. \end{aligned}$$

Since $q : U \rightarrow V$ is a \mathbb{P}^1 -bundle, we have $\dim U = \dim V + 1$. Therefore,

$$\dim V_c \geq \dim V - 1. \quad (2)$$

Step 1. We first show that if $f \not\subset E$ then $\sharp(\pi(f) \cap C) = 1$. Assume to the contrary that there exists $[f_0] \in V$ such that $f_0 \not\subset E$ and $\sharp(\pi(f_0) \cap C) \geq 2$. Let $a, b \in \pi(f_0) \cap C$ be distinct points. Note that $V_a \cap V_b \neq \emptyset$ since the point $[f_0]$ lies on the intersection. Using the inequality (2), we have

$$\dim(V_a \cap V_b) \geq \dim V_a + \dim V_b - \dim V = \dim V - 2 \geq 1.$$

Hence, there exists an irreducible curve $B \subset V_a \cap V_b$. Consider the ruled surface $S := q^{-1}(B)$. Since $\pi(f_0) \not\subset C$, the image $\pi \circ p(S)$ is a surface. We see that $U_a \cap S$ and $U_b \cap S$ are exceptional curves because these are respectively contracted to the points a and b . Thus, we have a contradiction by Lemma 1.

Step 2. Consider a rational curve f from the family V . Assume $f \not\subset E$. By Step 1, we have $\sharp(\pi(f) \cap C) = 1$. We put $c := \pi(f) \cap C$. By the inequality (2), there exists an irreducible curve B in V_c passing through the point $[f] \in V$. Consider the ruled surface $S := q^{-1}(B)$. Since $f \not\subset E$, we see that $p(S) \not\subset E$. We write

$$p^*E|_S = D + F$$

where D is the horizontal part and F is the vertical part, i.e. $\dim q(D) = 1$ and $\dim q(F) = 0$. We put $D' := \text{Supp}(D)$ and $F' := \text{Supp}(F)$. We have $D' \subset p^{-1}(E) \cap S$, hence $p(D') \subset E$. Recall that $B \subset V_c$. For any $v \in B \setminus q(F')$, we have $f_v \cap E_c \neq \emptyset$, and hence by Step 1, we see that $\pi(f_v) \cap C = c$, i.e. $f_v \cap E \subset E_c$. Let u be a point in $D' \setminus F'$. Since $q(u) \in B \setminus q(F')$, we have $f_{q(u)} \cap E \subset E_c$. Therefore,

$$p(u) \in f_{q(u)} \cap p(D') \subset f_{q(u)} \cap E \subset E_c.$$

Thus, $p(D' \setminus F') \subset E_c$. Taking the closure, we conclude that $p(D') \subset E_c$. Hence, $\pi \circ p(D') = c$. As in Step 1, we see that $\pi \circ p(S)$ is a surface. By Lemma 1, D' is a section of $q|_S$, which implies that $\sharp(E \cap f) = 1$. \square

3 Case of Fano manifolds with a divisorial contraction to a point

We first give an example in which the equality $i(X) = \ell(X)$ is easily verified. Let $\pi : X \rightarrow \mathbb{P}^n$ be the blow-up at a point a . We assume $n \geq 3$. We consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{P}^n \\ \varphi \downarrow & & \\ \mathbb{P}^{n-1} & & \end{array}$$

where $\varphi : X \rightarrow \mathbb{P}^{n-1}$ is the \mathbb{P}^1 -bundle whose fibers are the strict transforms of lines passing through a . Let e be a line in the exceptional divisor $E \simeq \mathbb{P}^{n-1}$ and let f be a fiber of φ . Then, $R_1 := \mathbb{R}^+[e]$ and $R_2 := \mathbb{R}^+[f]$ are extremal rays. Since $\rho(X) = 2$, we have

$$\overline{\text{NE}}(X) = R_1 + R_2.$$

Note that $\ell(R_1) = -K_X \cdot e = n - 1$ and $\ell(R_2) = -K_X \cdot f = 2$. Hence, we have $\ell(X) = 2$. Put $H := \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $L := \varphi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) = H - E$. Remark that L is the strict transform of a hyperplane containing the point a . We get

$$-K_X = \pi^*(-K_{\mathbb{P}^n}) - (n-1)E = (n+1)H - (n-1)(H-L) = 2H + (n-1)L.$$

Note that for a curve $\Gamma \subset X$, we have $\pi_* \Gamma \neq 0$ or $\varphi_* \Gamma \neq 0$. If Γ_0 is a rational curve such that $(-K_X) \cdot \Gamma_0 = i(X)$, then we have

$$i(X) = (-K_X) \cdot \Gamma_0 = (2H + (n-1)L) \cdot \Gamma_0 \geq 2 = \ell(X).$$

It follows that $i(X) = \ell(X)$.

We generalize the above example as follows:

Proposition 2. *Let X be a Fano manifold of dimension $n \geq 3$. Assume that there exists a birational extremal contraction $\pi : X \rightarrow Y$ sending a divisor to a point. Then, we have $i(X) = \ell(X)$.*

Proof. Let E be the exceptional divisor of π . By [20] Corollary 1.3, there exists an extremal ray $R \subset \overline{\text{NE}}(X)$ such that $E \cdot R > 0$, and the associated contraction $\varphi = \text{contr}_R : X \rightarrow Z$ is either:

1. a \mathbb{P}^1 -bundle,
2. a conic bundle with singular fibers, or
3. a smooth blow-up along a smooth subvariety of codimension 2.

In case (2) and (3), $\ell(R) = 1$. Hence, $i(X) = \ell(X) = 1$, and we are done. We assume that $\varphi : X \rightarrow Z$ is a \mathbb{P}^1 -bundle. Note that the base space Z is smooth (see [1]). Let f be a fiber of φ . Let B be an irreducible curve on Z passing through the point $\varphi(f)$. Consider the ruled surface $S := \varphi^{-1}(B)$. Since $S \cap E$ is an exceptional curve, by Lemma 1 it is a section of $\varphi|_S$. This implies that $E \cdot f = 1$ because the exceptional divisor E is reduced (see [7] Proposition 5-1-6). We conclude that $\varphi|_E : E \rightarrow Z$ is an isomorphism. Note that $E \simeq Z$ is smooth.

Using the rank 2 vector bundle $\mathcal{E} := \varphi_* \mathcal{O}_X(E)$, we write $X = \mathbb{P}(\mathcal{E})$. Pushing down the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(E) \rightarrow \mathcal{O}_E(E) \rightarrow 0,$$

we obtain

$$0 \rightarrow \varphi_* \mathcal{O}_X \rightarrow \varphi_* \mathcal{O}_X(E) \rightarrow \varphi_* \mathcal{O}_E(E) \rightarrow R^1 \varphi_* \mathcal{O}_X = 0.$$

Here, we have $\varphi_* \mathcal{O}_X \simeq \mathcal{O}_Z$ and $\varphi_* \mathcal{O}_E(E) \simeq \varphi_* N_{E/X}$. Thus, $\det \mathcal{E} \simeq \varphi_* N_{E/X}$. Note also that the hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is isomorphic to E . Using the canonical bundle formula for the \mathbb{P}^1 -bundle, we get

$$K_X = -2E + \varphi^*(K_Z + \varphi_* N_{E/X}). \quad (3)$$

Now, assume to the contrary that $\ell(X) > i(X)$, i.e. $\ell(X) = 2$ and $i(X) = 1$. Let $\Gamma_0 \subset X$ be a rational curve such that $-K_X \cdot \Gamma_0 = i(X) = 1$. In particular, Γ_0 is not a fiber of φ . Hence, $\varphi_* \Gamma_0 \neq 0$. If $\Gamma_0 \subset E$, then $-K_X \cdot \Gamma_0 \geq \ell(R) \geq \ell(X) = 2$, a contradiction. Thus, $\Gamma_0 \not\subset E$ so that $E \cdot \Gamma_0 \geq 0$. Since $\varphi : X \rightarrow Z$ is a \mathbb{P}^1 -bundle, by [15] Theorem 1.6 or [10] Corollary 2.9, we conclude that Z is a Fano manifold. In particular, $-K_Z \cdot \varphi_* \Gamma_0 > 0$. Note that the conormal bundle $N_{E/X}^*$ is ample, since E is an exceptional divisor. Therefore, using (3) we have

$$1 = -K_X \cdot \Gamma_0 = 2E \cdot \Gamma_0 + \varphi^*(-K_Z) \cdot \Gamma_0 + \varphi^*(\varphi_* N_{E/X}^*) \cdot \Gamma_0 \geq 0 + 1 + 1 = 2,$$

a contradiction. □

4 Classification results

In this section, we present results on a partial classification of Fano manifolds with $\ell(X) \geq 2$. These are used in the next section to prove our Theorem 1.

Lemma 2. *Let Y be a smooth projective variety of dimension $n \geq 4$. Let $\pi : X \rightarrow Y$ be the blow-up along a smooth curve $C \subset Y$. Assume that X is a Fano manifold. Let E be the exceptional divisor of π . Then, there exists an extremal ray $R \subset \overline{NE}(X)$ such that $E \cdot R > 0$. Furthermore, every non-trivial fiber of the associated contraction $\varphi : X \rightarrow Z$ has dimension at most 2.*

Proof. This follows from a similar argument as in [3] Section 2. \square

Throughout the section, we fix the notation of this lemma.

Proposition 3. *If φ is a fiber type contraction with $\dim Z = n - 2$ and $\ell(R) \geq 2$, then we have either: $Y = \mathbb{P}^n$ and C is a line, or $Y = Q_n$ and C is a conic not on a plane contained in Q_n .*

Proof. Since $\ell(R) \geq 2$, the general fiber of φ is isomorphic to \mathbb{P}^2 or Q_2 . Hence, the statement follows from [17] Theorem 1.1. \square

If the general fiber of φ has dimension one, we have $\ell(R) \leq 2$. We prove the following:

Proposition 4. *If φ is a fiber type contraction with $\dim Z = n - 1$ and $\ell(R) = 2$, then the pair (Y, C) is exactly one of the following:*

1. $Y = Q_n$ and C is a line;
2. $Y = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ and C is a fiber of the projection $\mathbb{P}^1 \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$;
3. $Y = \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$ and C is the strict transform of a line in \mathbb{P}^n ;
4. $Y = \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$ and C is a fiber of the exceptional divisor;
5. $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)})$ and C is the section such that $N_{C/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n-1)}$.

Proof. Let $f \simeq \mathbb{P}^1$ be a general fiber of φ such that $f \not\subset E$. Since $\ell(R) = 2$, the family of rational curves containing the point $[f]$ is unsplit. By Proposition 1, we have $\sharp(E \cap f) = 1$. This implies $E \cdot f = 1$ because the exceptional divisor E is reduced.

We first consider the case where the restriction map $\varphi|_E : E \rightarrow Z$ is not finite, i.e. there exists a curve \tilde{C} contained in E and contracted by φ . By definition of an extremal contraction, there exists $b \in \mathbb{R}^+$ such that $\tilde{C} \equiv bf$. Recall that E is a \mathbb{P}^{n-2} -bundle over the curve C . Since \tilde{C} is an exceptional curve, its numerical class generates an extremal ray. Since $\rho(E) = 2$, we have

$$\overline{NE}(E) = \mathbb{R}^+[\tilde{C}] + \mathbb{R}^+[e]$$

where e is a line in a fiber of $\pi|_E : E \rightarrow C$. Using the adjunction formula: $K_E = (K_X + E)|_E$ and the equality $E \cdot f = 1$, we get

$$-K_E \cdot e = -K_X \cdot e - E \cdot e = (n - 2) + 1 = n - 1 > 0$$

and

$$-K_E \cdot \tilde{C} = b(-K_X \cdot f - E \cdot f) = b(2 - 1) = b > 0. \quad (4)$$

By Kleiman's criterion, $-K_E$ is ample. Since the Fano manifold E is a \mathbb{P}^{n-2} -bundle over a curve and contains an exceptional curve, it is isomorphic to $\text{Bl}_{\mathbb{P}^{n-3}}(\mathbb{P}^{n-1})$. Since $E \cdot f = 1$, we see that $\varphi|_E : E \rightarrow Z$ is generically one to one onto the normal variety Z . Hence, the finite part of its

Stein factorization is an isomorphism. It follows that $\varphi|_E : E \rightarrow Z$ coincides with the blow-up $\text{Bl}_{\mathbb{P}^{n-3}}(\mathbb{P}^{n-1}) \rightarrow \mathbb{P}^{n-1}$. Hence, Z is isomorphic to \mathbb{P}^{n-1} . We have also $\rho(X) = \rho(Z) + 1 = 2$ and $\rho(Y) = 1$. We observe that φ_*e is a line in $Z \simeq \mathbb{P}^{n-1}$. So, if we put $L := \varphi^*\mathcal{O}_Z(1)$, then $L \cdot e = 1$. Since $-K_X \cdot e = n - 2$ and $-K_X \cdot f = 2$, we have $-K_X = nL + 2E$. On the other hand,

$$-K_X = \pi^*(-K_Y) - (n - 2)E = r(Y)H - (n - 2)E$$

where $r(Y)$ is the index of Y and H is the pull back by π of the ample generator of $\text{Pic}(Y) \simeq \mathbb{Z}$. Thus, we have

$$r(Y)H = n(L + E).$$

Since $(L + E) \cdot e = 0$, there exists $D \in \text{Pic}(Y)$ such that $L + E = \pi^*D$. Note that we can write $\pi^*D = dH$ with $d \in \mathbb{Z}$. Hence, $r(Y) = nd$. By [8], we have $r(Y) = n$ and Y is isomorphic to Q_n . We have also $H \cdot f = 1$. Now, we know that the curve \tilde{C} defined above is a fiber of the exceptional divisor of the blow-up $E \simeq \text{Bl}_{\mathbb{P}^{n-3}}(\mathbb{P}^{n-1}) \rightarrow \mathbb{P}^{n-1}$. Since $-K_E \cdot \tilde{C} = 1$, we get $b = 1$ from (4), and we see that \tilde{C} is numerically equivalent to f . Hence, we have

$$\mathcal{O}_Y(1) \cdot C = H \cdot \tilde{C} = H \cdot f = 1.$$

It follows that C is a line in $Y \simeq Q_n$. Hence, we get the example (1).

Now, we consider the case where $\varphi|_E : E \rightarrow Z$ is finite. Note that every fiber of φ is one-dimensional. Hence, by the assumption that $\ell(R) = 2$, we see that φ is a \mathbb{P}^1 -bundle (see [1] and [20] Theorem 1.2). By [15] Theorem 1.6 or [10] Corollary 2.9, we conclude that Z is a Fano manifold. Note that $\varphi|_E$ is an isomorphism because $E \cdot f = 1$. Since $Z \simeq E$ has a structure of a \mathbb{P}^{n-2} -bundle over the curve C , Z is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ or $\text{Bl}_{\mathbb{P}^{n-3}}(\mathbb{P}^{n-1})$, and C is isomorphic to \mathbb{P}^1 .

Claim 1. Y is a \mathbb{P}^{n-1} -bundle over \mathbb{P}^1 .

Proof. For a point $a \in C$, we put $E_a := \pi^{-1}(a)$, $Z_a := \varphi(E_a)$, $X_a := \varphi^{-1}(Z_a)$ and $Y_a := \pi(X_a)$. Note that X_a is smooth because it is a \mathbb{P}^1 -bundle over $Z_a \simeq \mathbb{P}^{n-2}$. Hence, the divisor $Y_a \subset Y$ is smooth in codimension one. Thus, Y_a is normal ([5] Ch. II Proposition 8.23). Note that $N_{E_a/X_a} \simeq \mathcal{O}_{\mathbb{P}^{n-2}}(-1)$. It follows that $\pi|_{X_a} : X_a \rightarrow Y_a$ is the blow-up at the point $C \cap Y_a$ with the exceptional divisor E_a . On the other hand, $\varphi|_{X_a} : X_a \rightarrow Z_a \simeq \mathbb{P}^{n-2}$ is a \mathbb{P}^1 -bundle. Consider the composite map $\Phi : X \rightarrow Z \simeq E \rightarrow C \simeq \mathbb{P}^1$. The fiber $X_a = \Phi^{-1}(a)$ is a Fano manifold. So, by the classification result due to [3], we conclude that Y_a is isomorphic to \mathbb{P}^{n-1} . Consider the nef divisor $F := \Phi^*\mathcal{O}_{\mathbb{P}^1}(1)$. We see that $F - K_X$ is ample. Hence, $H^1(X, \mathcal{O}_X(F)) = 0$. From the exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_X(F + E) \rightarrow \mathcal{O}_E(F + E) \rightarrow 0,$$

we get

$$h^0(X, \mathcal{O}_X(F + E)) \geq h^0(X, \mathcal{O}_X(F)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2.$$

Let M be a general member of $|F + E|$ and we put $M' := \pi(M)$. Since $M \cdot f = (F + E) \cdot f = E \cdot f = 1$, we have $M'|_{Y_a} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Hence the morphism $\Phi \circ \pi^{-1} : Y \rightarrow \mathbb{P}^1$ is a \mathbb{P}^{n-1} -bundle and C is a section. \square

If Y is a Fano manifold, Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ or $\text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$. We first treat the case $Y \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$. Assume that C is not a fiber of the projection $pr : \mathbb{P}^1 \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-2}$. Let Γ be a fiber of pr meeting C and $\tilde{\Gamma}$ its strict transform by the blow-up $\pi : X \rightarrow Y$. Then, we have

$$K_X \cdot \tilde{\Gamma} = K_Y \cdot \Gamma + (n - 2)E \cdot \tilde{\Gamma} \geq -2 + (n - 2) = n - 4 \geq 0,$$

which contradicts the assumption that X is a Fano manifold. It follows that C is a fiber of the projection pr and we get the example (2). Now, we consider the case $Y \simeq \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$. Let G

be the exceptional divisor of the blow-up $\beta : \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n) \rightarrow \mathbb{P}^n$. Assume that $G \cdot C > 0$. Let g be a fiber of the \mathbb{P}^1 -bundle $G \rightarrow \mathbb{P}^{n-2}$ such that $g \cap C \neq \emptyset$. Then, we have

$$K_X \cdot \tilde{g} = K_Y \cdot g + (n-2)E \cdot \tilde{g} \geq -1 + (n-2) = n-3 > 0,$$

a contradiction. Hence, $G \cdot C \leq 0$. Since C is a section of the \mathbb{P}^{n-1} -bundle $Y \rightarrow \mathbb{P}^1$, C is either the strict transform by β of a line in \mathbb{P}^n which does not meet the center \mathbb{P}^{n-2} , or a fiber of the \mathbb{P}^1 -bundle $G \rightarrow \mathbb{P}^{n-2}$. Thus, we get the examples (3) and (4).

If Y is not a Fano manifold, by [20] Proposition 3.5, we have $N_{C/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(n-1)}$. Hence, we conclude that $Y \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)})$ and we get the example (5). \square

Remarks. In dimension three, there is another example: $Y = \mathbb{P}^3$ and C is a rational curve of degree 3 (see [13] n° 27 in Table 2). If we assume $i(X) \geq 2$, a similar statement is derived from [2] Theorem 1.3. In dimension four, if we assume $\rho(X) = 2$ and $\varphi : X \rightarrow Z$ is a *scroll in the sense of adjunction theory* (see [4] for the definition), the example $(Y, C) = (Q_4, \text{line})$ is obtained from the list in [11].

5 Proof of Theorem 1

Let X be a smooth Fano 4-fold. Let $R \subset \overline{\text{NE}}(X)$ be the extremal ray defining the birational contraction $\pi : X \rightarrow Y$. If $\ell(R) = 1$, then $\ell(X) = i(X) = 1$, hence we are done. So, it suffices to consider the case $\ell(R) \geq 2$. Recall that a contraction $\pi : X \rightarrow Y$ is called (a, b) -type if $\dim \text{Exc}(\pi) = a$ and $\dim \pi(\text{Exc}(\pi)) = b$. By Fujita-Ionescu-Wisniewski's inequality (see [6] Theorem 0.4 and [20] Theorem 1.1):

$$\dim(\text{non-trivial fiber of } \pi) + \dim \text{Exc}(\pi) \geq \dim X + \ell(R) - 1,$$

we conclude that $\pi : X \rightarrow Y$ is either of type (3,0) or (3,1). By Proposition 2, we have $i(X) = \ell(X)$ in the case of (3,0)-type.

Now, assume that $\pi : X \rightarrow Y$ is a (3,1)-type contraction. Since $\ell(R) \geq 2$, π is a blow-up along a smooth curve C and Y is smooth (see [16]). Let E be the exceptional divisor of π . By Lemma 2, there exists an extremal ray $R' \subset \overline{\text{NE}}(X)$ such that $E \cdot R' > 0$. Let $\varphi : X \rightarrow Z$ be the associated contraction. Recall that the fiber of φ has dimension at most 2. Since $\ell(R') \geq \ell(X) \geq 2$, φ is one of the following:

1. (3,1)-type: blow-up along a smooth curve and Z is smooth;
2. (4,2)-type: the general fiber of φ is isomorphic to \mathbb{P}^2 or Q_2 ;
3. (4,3)-type: the general fiber of φ is isomorphic to \mathbb{P}^1 and forms an unsplit family.

The case (1) is excluded by [18] Proposition 5. In the case (2), by Proposition 3, we have $Y = \mathbb{P}^4$ and C is a line, or $Y = Q_4$ and C is a conic (not on a plane contained in Q_4). Concerning the case (3), note that among 5 examples in Proposition 4, the condition $\ell(X) \geq 2$ is satisfied only for the examples (1) and (2). Summing up, we have the following possibilities:

1. $Y = \mathbb{P}^4$ and C is a line;
2. $Y = Q_4$ and C is a conic;
3. $Y = Q_4$ and C is a line;
4. $Y = \mathbb{P}^1 \times \mathbb{P}^3$ and C is a fiber of the projection $\mathbb{P}^1 \times \mathbb{P}^3 \rightarrow \mathbb{P}^3$.

Hence, to prove Theorem 1, it is sufficient to verify that the equality $i(X) = \ell(X)$ holds for $X = \text{Bl}_C(Y)$ in the above four examples. We just check the case (1), since the argument is similar in the other cases. Let C be a line in \mathbb{P}^4 . Note that $X = \text{Bl}_C(\mathbb{P}^4)$ has two extremal contractions: the blow-up $\pi : X \rightarrow \mathbb{P}^4$ along C and the \mathbb{P}^2 -bundle $\varphi : X \rightarrow \mathbb{P}^2$. The exceptional divisor $E = \text{Exc}(\pi)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ and the restrictions $\pi|_E : E \rightarrow C \simeq \mathbb{P}^1$ and $\varphi|_E : E \rightarrow \mathbb{P}^2$ are the two natural projections. Let e be a line in a fiber of $\pi|_E$ and f be a fiber of $\varphi|_E$. Then, we have $\overline{\text{NE}}(X) = \mathbb{R}^+[e] + \mathbb{R}^+[f]$. Note that f is numerically equivalent to the strict transform by π of a line meeting C . Since $-K_X \cdot e = 2$ and $-K_X \cdot f = 3$, we have $\ell(X) = 2$. Let Γ be an irreducible curve on X . Assume that $\Gamma \not\subset E$. Let H be the pull back by π of a hyperplane containing the line C but not containing the curve $\pi(\Gamma)$. Then, we have $H \cdot \Gamma \geq E \cdot \Gamma$. Hence,

$$-K_X \cdot \Gamma = (\pi^*(-K_{\mathbb{P}^4}) - 2E) \cdot \Gamma = (5H - 2E) \cdot \Gamma \geq 3H \cdot \Gamma \geq 3.$$

If $\Gamma \subset E \simeq \mathbb{P}^1 \times \mathbb{P}^2$, there exists natural numbers a and b such that $\Gamma \equiv ae + bf$. We have $-K_E \cdot \Gamma = 3a + 2b$, and $E|_E \cdot \Gamma = aE \cdot e + bE \cdot f = -a + b$. Hence,

$$-K_X \cdot \Gamma = (-K_E + E|_E) \cdot \Gamma = (3a + 2b) + (-a + b) = 2a + 3b \geq 2.$$

Therefore, $i(X) \geq 2$. Thus, $i(X) = \ell(X) = 2$. □

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References

- [1] T. Ando, On extremal rays of the higher-dimensional varieties, *Invent. Math.* **81**, 347–357 (1985)
- [2] M. Andreatta, G. Occhetta, Fano manifolds with long extremal rays. *Asian J. Math.* **9**, 523–543 (2005)
- [3] L. Bonavero, F. Campana, J. A. Wiśniewski, Variétés complexes dont l'éclatée en un point est de Fano. *C. R. Math. Acad. Sci. Paris Ser. I* **334**, 463–468 (2002)
- [4] M. Beltrametti, A. Sommese, The adjunction theory of complex projective varieties. de Gruyter Expositions in Mathematics, 16. Walter de Gruyter, Berlin, (1995)
- [5] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg (1977)
- [6] P. Ionescu, Generalized adjunction and applications. *Math. Proc. Cambridge Philos. Soc.* **99**, 457–472 (1986)
- [7] Y. Kawamata, K. Matsuda, K. Matsuki, Introduction to the minimal model problem. Algebraic geometry, Sendai, 1985, 283–360, *Adv. Stud. Pure Math.*, 10, North-Holland, Amsterdam (1987)
- [8] S. Kobayashi, T. Ochiai, Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.* **13**, 31–47 (1973)
- [9] J. Kollár, Rational Curves on Algebraic Varieties, volume 32 of *Ergebnisse der Math.* Springer Verlag (1996)
- [10] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.* **36**, 765–779 (1992)
- [11] A. Langer, Fano 4-folds with scroll structure. *Nagoya Math. J.* **150**, 135–176 (1998)
- [12] S. Mori, Projective manifolds with ample tangent bundles, *Ann. of Math.* **110**, 593–606 (1979)
- [13] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$, *Manuscripta Math.* **36**, 147–162 (1981/82)
Erratum: *Manuscripta Math.* **110**, 407 (2003)

- [14] T. Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, 15. Springer-Verlag, Berlin, (1988)
- [15] M. Szurek, J. A. Wiśniewski, Fano bundles over \mathbb{P}^3 and Q_3 . *Pacific J. Math.* **141**, 197–208 (1990)
- [16] H. Takagi, Classification of extremal contractions from smooth fourfolds of (3,1)-type, *Proc. Amer. Math. Soc.* **127**, 315–321 (1999)
- [17] T. Tsukioka, Del Pezzo surface fibrations obtained by blow-up of a smooth curve in a projective manifold. *C. R. Acad. Sci. Paris, Ser. I* **340**, 581–586, (2005)
- [18] T. Tsukioka, A remark on Fano 4-folds having (3,1)-type extremal contractions. *Math. Ann.*, to appear.
- [19] J. A. Wiśniewski, On a conjecture of Mukai, *Manuscripta Math.* **68**, 135–141 (1990)
- [20] J. A. Wiśniewski, On contractions of extremal rays of Fano manifolds. *J. Reine Angew. Math.* **417**, 141–157 (1991)

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